Spin-wave damping in itinerant electron ferromagnets

1990 J. Phys.: Condens. Matter 2 3579

(http://iopscience.iop.org/0953-8984/2/15/014)

J. Phys.: Condens. Matter **2** (1990) 3579–3594. Printed in the UK

Spin-wave damping in itinerant electron ferromagnets

M Isoda
Department of Physics, Faculty of Liberal Arts, Fukuyama University, Fukuyama, Japan

Received 3 May 1989, in final form 3 July 1989

Abstract. The damping or the lifetime of long-wavelength spin waves in itinerant electron ferromagnets is evaluated with the Hubbard Hamiltonian. The spin-wave pole is derived by using the diagrams for the irreducible susceptibility given by Ma *et al.* It is shown that even for weak ferromagnets the damping due to the scattering process of the spin wave is important, especially at low temperatures, in contrast to previous expectations from the results for the nearly ferromagnetic case under an external field. Perfect ferromagnets are also considered. The spin wave in weak ferromagnets is more strongly damped than that in the perfect ones by order of ζ^{-4}, where ζ is the relative magnetisation.

1. Introduction

Many studies on spin waves in itinerant electron ferromagnets have been performed theoretically and experimentally. Of major interest has been the spin-wave energy. On the other hand, the damping, which is inversely proportional to the lifetime, of spin waves has been investigated by only a few groups. The reason seems to be a lack of experimental data due to the difficulty of neutron scattering experiments for the spin-wave damping.

Experimentally, as far as the author is aware, only one such study was presented for Fe and Ni (Stringfellow 1968) up to 1980, except for typical Heisenberg ferromagnets, and he concluded that the results are consistent with the theoretical results for the magnon–magnon interaction in the Heisenberg model. However, recent investigations for invar alloys (Onodera *et al* 1981) have proposed very strong damping remarkably different from Fe and Ni.

On the other hand, theoretical investigations for the spin-wave damping in itinerant electron ferromagnets have been performed for two mechanisms: electron–phonon (Yamada 1976) and electron correlation, briefly reviewed below. However, the previous theories based on both mechanisms did not derive the strong damping detected in invar alloys. It is still unknown whether the origin of this strong damping is due to invar characteristics or to general electronic properties. Therefore, with the expectation of experiments in various itinerant ferromagnets in future, the development of the electron correlation theory for spin-wave damping seems to be meaningful.

Up to now, only three theoretical investigations on these lines have been performed. The first is by Thompson (1965), who calculated the intrinsic (temperature-independent)

0953-8984/90/153579 + 16 $03.50

lifetime by the second-order perturbation calculation. The other two used the diagrammatic perturbation method and derived the temperature dependence of the damping. One of them is by Ma *et al* (1968) for ^{3}He under an external magnetic field, and the other is by Gergis (1972) for perfect ferromagnets. The former only takes into account the decay process of the spin wave to the paramagnons, i.e. the final states of the spin wave are treated as the paramagnetic state without the external field. Therefore this theory may be appropriate for the nearly ferromagnetic case under an external magnetic field where the spin-wave dispersion has a gap, though it seems to be doubtful whether the theory is applicable to weak ferromagnets, in contrast to the expectation of Edwards and Fisher (1971). The reason is that because of the absence of a gap in the weakly ferromagnetic case without an external field, it is expected to be important for the scattering process of the referred spin wave into the spin wave and the longitudinal fluctuations, especially at low temperature.

Thus it is now necessary to investigate *real* weak ferromagnets to understand the spin-wave damping in comparison with perfect ferromagnets. Then the present purpose is to investigate the damping of the spin wave with long wavelength in both limits of perfect and weak ferromagnets, with special interest in its temperature dependence.

In the next section, the expression of the transverse dynamical susceptibility is derived and in section 3 a general expression of the spin-wave damping within the scope of the approximation in section 2 is given. The spin-wave damping is calculated for the limits of weak and perfect ferromagnets in sections 4 and 5, respectively. The last section is devoted to the conclusions and a discussion.

2. Transverse dynamical susceptibility

We start from the Hubbard Hamiltonian with an intra-atomic Coulomb interaction I,

$$H = \sum_k \sum_\sigma \varepsilon_{k\sigma} a^\dagger_{k\sigma} a_{k\sigma} + I \sum_{kk'} \sum_q a^\dagger_{k+q\uparrow} a_{k\uparrow} a^\dagger_{k'-q\downarrow} a_{k'\downarrow}$$
$$\varepsilon_{k\sigma} = \varepsilon_k - \sigma B \tag{2.1}$$

where $a_{k\sigma}$ ($a^\dagger_{k\sigma}$) is an annihilation (creation) operator of an electron with wavevector k and spin σ. The one-electron energy and the exchange field are denoted by ε_k and B, respectively. The transverse dynamical susceptibility is given by the following standard formula:

$$\chi^{-+}(\boldsymbol{Q}) = \sum_k \chi^{-+}(k, \boldsymbol{Q})$$
$$\chi^{-+}(k, \boldsymbol{Q}) = \int_0^{1/T} \mathrm{d}\tau\, \mathrm{e}^{\tau Q_0} \langle T_\tau S^-_{k,\boldsymbol{Q}}(\tau) S^+_{-\boldsymbol{Q}}(0)\rangle \tag{2.2}$$

with

$$S^+_{\boldsymbol{Q}} = \sum_k S^+_{k,\boldsymbol{Q}} = \sum_k a^\dagger_{k\uparrow} a_{k+Q\downarrow} \tag{2.3}$$

where $\boldsymbol{Q} = (Q, Q_0)$ is a four-component vector, Q the wavevector, $Q_0 = 2m\pi \mathrm{i} T$ (m = integer), T_τ the imaginary time ordering operator and the temperature T is expressed in energy units.

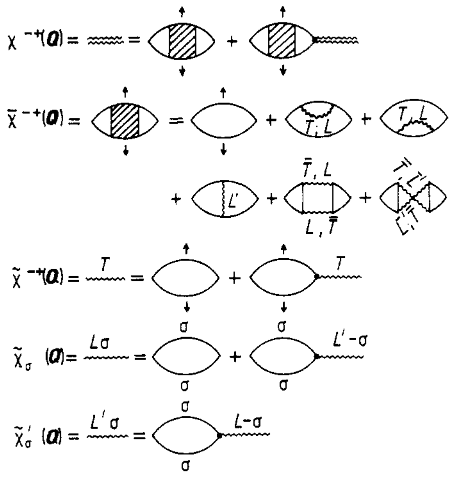

Figure 1. Diagrammatic equation of the transverse dynamical susceptibility and of the irreducible one. $\bar{T}$ and $\bar{L}'$ mean taking a bare interaction in addition to the corresponding spin fluctuation propagators.

Now we introduce the irreducible susceptibility $\bar{\chi}^{-+}(Q)$ defined by sets of bubble diagrams each of which cannot be separated into two pieces, each having an external vertex, by removing an interaction vertex. Then the dynamical susceptibility is given by

$$\chi^{-+}(Q) = \bar{\chi}^{-+}(Q)/[1 - I\bar{\chi}^{-+}(Q)] \tag{2.4}$$

and is shown diagrammatically in figure 1.

For the irreducible diagrams various choices are possible under the condition of total spin conservation of the Hamiltonian. Ma *et al* (1968), Hertz (1971) and Kawabata (1974) approximated the irreducible susceptibilities by a set of bubble diagrams including a finite number of internal spin fluctuation propagators. On the other hand, Moriya (1976) used the self-consistent expression having infinite number of spin fluctuation propagators in the irreducible diagrams for the purpose of discussing the persistence of spin waves above the Curie temperature.

In the present problem of describing the damping of a long-wavelength spin wave, the former approximation seems to be sufficient. The expression for the irreducible transverse susceptibility is given diagrammatically in figure 1. Here the internal spin fluctuation propagators are approximated by the random-phase approximation (RPA) and a full line corresponds to the one-electron Green function,

$$G_\sigma(\boldsymbol{k}) = (k_0 - \varepsilon_{\boldsymbol{k}\sigma} + \mu)^{-1} \tag{2.5}$$

where μ is the chemical potential, $k_0 = (2m+1)\pi \mathrm{i} T$ (m = integer). On the present choice of the irreducible susceptibility, only the most simple interaction processes are taken into account, where the referred spin wave is transferred into two modes, one transverse and the other longitudinal, because the many fluctuation processes seem to be less important at the temperature concerned. Six bubble diagrams within the irreducible susceptibility in figure 1 are denoted by $\Lambda_i(Q)$, $i = 1, \ldots, 6$, in order and are expressed as follows:

$$\bar{\chi}^{-+}(Q) = \Lambda_1(Q) + \delta\bar{\chi}^{-+}(Q) \qquad \delta\bar{\chi}^{-+}(Q) = \sum_{i=2}^{6} \Lambda_i(Q)$$

$$\Lambda_1(Q) = \chi_0^{-+}(Q) = -T\sum_{\boldsymbol{k}} G_\uparrow(\boldsymbol{k})G_\downarrow(\boldsymbol{k}+Q)$$

$$\Lambda_2(Q) = -T\sum_k \frac{\Delta n_\uparrow(k)}{\xi(k,Q)} + I^2T\sum_{k,q}\frac{1}{[\xi(k,Q)]^2}[\tilde{\chi}^{-+}(q)\Gamma_\downarrow - \tilde{\chi}_\downarrow(Q-q)\Gamma_\uparrow]$$

$$\Lambda_3(Q) = T\sum_k \frac{\Delta n_\downarrow(k)}{\xi(k-Q,Q)} + I^2T\sum_{k,q}\frac{1}{[\xi(k,Q)]^2}[\tilde{\chi}^{-+}(q)\Gamma_\uparrow - \tilde{\chi}_\uparrow(Q-q)\Gamma_\downarrow] \tag{2.6}$$

$$\Lambda_4(Q) = I^2T\sum_{k,q}\frac{1}{\xi(k,Q)}\left(\frac{\tilde{\chi}'_\uparrow(Q-q)}{\xi(k+q-Q,Q)}\Gamma_\downarrow + \frac{\tilde{\chi}'_\downarrow(Q-q)}{\xi(k-q+Q,Q)}\Gamma_\uparrow\right)$$

$$\Lambda_5(Q) = I^3T\sum_q[1+I\tilde{\chi}^{-+}(q)]\sum_\sigma\tilde{\chi}_\sigma(Q-q)\left(\sum_k\frac{\Gamma_{-\sigma}}{\xi(k,Q)}\right)^2,$$

$$\Lambda_6(Q) = -I^2T\sum_q[1+I\tilde{\chi}^{-+}(q)]\left(I\sum_\sigma\tilde{\chi}'_\sigma(Q-q)-2\right)\prod_\sigma\sum_k\frac{\Gamma_\sigma}{\xi(k,Q)}$$

with

$$\begin{aligned}\Gamma_\uparrow &= \chi_{0\uparrow}(k,Q-q) - \chi_0^{-+}(k+Q-q,q)\\ \Gamma_\downarrow &= \chi_{0\downarrow}(k+q,Q-q) - \chi_0^{-+}(k,q)\\ \xi(k,Q) &= Q_0 + \varepsilon_{k\uparrow} - \varepsilon_{k+Q\downarrow}\\ \Delta n_\sigma(k) &= I^2T[G_\sigma(k)]^2\sum_q[\tilde{\chi}^{-\sigma\sigma}(q)G_{-\sigma}(k+q) + \tilde{\chi}_{-\sigma}(q)G_\sigma(k+q)]\end{aligned} \tag{2.7}$$

by the use of the identity for the Green functions,

$$G_\sigma(k)G_{\sigma'}(k+q) = [G_\sigma(k) - G_{\sigma'}(k+q)]/\xi(k,q) \tag{2.8}$$

and of the symmetry relations,

$$\begin{aligned}\chi^{-+}(Q) &= \chi^{+-}(-Q)\\ \chi_\sigma(Q) &= \chi_\sigma(-Q)\\ \chi'_\sigma(Q) &= \chi'_{-\sigma}(-Q).\end{aligned} \tag{2.9}$$

The internal RPA spin fluctuation propagators are given by

$$\begin{aligned}\tilde{\chi}^{-+}(q) &= \chi_0^{-+}(q)/[1 - I\chi_0^{-+}(q)]\\ \tilde{\chi}_\sigma(q) &= \chi_{0\sigma}(q)/[1 - I^2\chi_{0\sigma}(q)\chi_{0-\sigma}(q)]\\ \tilde{\chi}'_\sigma(q) &= -I\chi_{0\sigma}(q)\chi_{0-\sigma}(q)/[1 - I^2\chi_{0\sigma}(q)\chi_{0-\sigma}(q)]\end{aligned} \tag{2.10}$$

where $\chi_0(q)$ are the non-interacting susceptibilities and are expressed in appendix 1.

3. Spin-wave damping

Now we may obtain the spin-wave energy and the damping from the real and imaginary parts, respectively, of the pole of the transverse dynamical susceptibility in equation (2.4). Therefore the equation to be solved is

$$1 - I\bar{\chi}^{-+}(Q) = 0 \tag{3.1}$$

and the damping is then proportional to the imaginary part of $\bar{\chi}^{-+}(Q)$.

Before performing the practical calculation of equation (2.6), the most simple approximation is noticed. That is the RPA theory for $\chi^{-+}(Q)$, which corresponds to approximating the irreducible susceptibility only by the first bubble, $\Lambda_1(Q)$. In this approximation the spin wave naturally does not damp and is well defined in the vanishing region of the imaginary part of non-interacting transverse susceptibility $\chi_0^{-+}(Q)$ (see figure 2 in appendix 1) at absolute zero temperature.

In the present approximation, the imaginary parts of the irreducible susceptibility for the long-wavelength modes come from the last five bubble diagrams, i.e. Λ_i ($i = 2, \ldots, 6$). The first terms within Λ_2 and Λ_3 contribute to the shift of the chemical potential; therefore, those terms are absorbed in μ. Then we redefine Λ_2 and Λ_3 by respective second terms. Thus redefined $\delta\bar{\chi}^{-+}(Q)$ vanishes in the limit of long wavelength, i.e. $Q \to 0$, as the requirement of total spin conservation (Ma *et al* 1968). Therefore $\delta\bar{\chi}^{-+}(Q)$ is expanded by Q to second order. After performing the summation on q_0 and then replacing Q_0 by $\Omega + \mathrm{i}s$ (s = positive infinitesimal), the shift of the spin-wave energy, or the correction for the spin-wave stiffness constant, and the damping may be calculated from the real and imaginary parts of $\delta\bar{\chi}^{-+}(Q, \Omega + \mathrm{i}s)$, respectively. However, since our present interest is in the damping, only the imaginary part is calculated.

Then the results are expressed by the use of the electron-gas model and of the reduced quantities, the wavevector and the energy normalised by the Fermi vector k_F and the Fermi energy ε_F in the paramagnetic state, respectively, as follows:

$$
\begin{aligned}
&\mathrm{Im}[I\delta\bar{\chi}^{-+}(Q, \Omega)] = \eta_1 + \eta_2 + \eta_3 \\
&\eta_1 = \frac{4Q^2\alpha^2}{3\xi_0^4}\int \mathrm{d}\omega\, N(\omega, \Omega)\int_0^{q_c} \mathrm{d}q\, q^2\, \mathrm{Im}\,\tilde{T}\,[\mathrm{Im}\,J_2(-q, \Omega - \omega) \\
&\qquad + \mathrm{Im}\,J_4(-q, \Omega - \omega)] \\
&\eta_2 = -\frac{4Q^2\alpha^3}{3\pi\xi_0^4}\int \mathrm{d}\omega\, N(\omega, \Omega)\int_0^{q_c} \mathrm{d}q\,\frac{q^2}{l} \\
&\qquad \times \{\pi[\psi_\uparrow\, \mathrm{Im}\,J_1(q, \omega) + \psi_\downarrow\, \mathrm{Im}\,J_3(q, \omega)] + 2\alpha\lambda_1 q^2\, \mathrm{Im}\, f_0^{-+}\} \\
&\eta_3 = \frac{2Q^2\alpha^3}{3\pi\xi_0^4}\int \mathrm{d}\omega\, N(\omega, \Omega)\int_0^{q_c} \mathrm{d}q \sum_\sigma [\![2\,\mathrm{Re}\,\tilde{T}\,\mathrm{Im}\, f_0^{-+} \\
&\qquad \times [\mathrm{Re}\,\tilde{L}\,\mathrm{Im}\, f_{0\sigma}(r_\sigma\tilde{r}_{-\sigma} - \sigma\alpha r_\sigma\varphi_\sigma - \alpha r_{-\sigma}\tilde{r}_{-\sigma}\,\mathrm{Re}\, f_{0-\sigma}) \\
&\qquad + \mathrm{Im}\,\tilde{L} r_\sigma(\sigma\varphi_{-\sigma} + \sigma\alpha\varphi_\sigma\,\mathrm{Re}\, f_{0\sigma} + \alpha\tilde{r}_\sigma\,\mathrm{Im}\, f_{0\sigma}\,\mathrm{Im}\, f_{0-\sigma})] \\
&\qquad + \mathrm{Im}\,\tilde{T}[\![\mathrm{Re}\,\tilde{L}\,\mathrm{Im}\, f_{0\sigma}\{2\sigma\tilde{r}_{-\sigma}(\varphi_\sigma + \alpha\varphi_{-\sigma}\,\mathrm{Re}\, f_{0-\sigma}) \\
&\qquad + \alpha[\varphi_\sigma^2 - r_\sigma^2(\mathrm{Im}\, f_0^{-+})^2 - \tilde{r}_\sigma^2(\mathrm{Im}\, f_{0-\sigma})^2]\} \\
&\qquad + \mathrm{Im}\,\tilde{L}\{\varphi_\sigma\varphi_{-\sigma} + r_\sigma r_{-\sigma}(\mathrm{Im}\, f_0^{-+})^2 \\
&\qquad + \tilde{r}_\sigma(\tilde{r}_{-\sigma} + 2\sigma\alpha\varphi_\sigma)\,\mathrm{Im}\, f_{0\sigma}\,\mathrm{Im}\, f_{0-\sigma} \\
&\qquad + \alpha\,\mathrm{Re}\, f_{0\sigma}[\varphi_\sigma^2 - r_\sigma^2(\mathrm{Im}\, f_0^{-+})^2 - \tilde{r}_\sigma^2(\mathrm{Im}\, f_{0-\sigma})^2]\}]\!]]\!]
\end{aligned}
\qquad (3.2)
$$

with

$$N(\omega, \Omega) = \mathrm{sgn}(\omega)[\tfrac{1}{2} + n(|\omega|)] - \mathrm{sgn}(\omega - \Omega)[\tfrac{1}{2} + n(|\omega - \Omega|)]$$

$$\tilde{T} = (1 - \alpha f_0^{-+})^{-1} \qquad \tilde{L} = (1 - \alpha^2 f_{0\uparrow} f_{0\downarrow})^{-1}$$

$$
\begin{aligned}
&l = (1 - \alpha^2\lambda_R)^2 + \alpha^4\lambda_I^2 \\
&\psi_\sigma = \mathrm{Im}\, f_{0\sigma} + \alpha\lambda_I + \alpha^2\, \mathrm{Im}\, f_{0-\sigma}[(\mathrm{Re}\, f_{0\sigma})^2 + (\mathrm{Im}\, f_{0\sigma})^2] \\
&\varphi_\sigma = \tfrac{2}{3}(k_\downarrow^3 - k_\uparrow^3) + \sigma r_\sigma\, \mathrm{Re}\, f_0^{-+} - \sigma\tilde{r}_\sigma\, \mathrm{Re}\, f_{0-\sigma} \\
&r_\sigma = 2q\sigma P_\sigma \\
&\tilde{r}_\sigma = 2q\sigma\tilde{P}_\sigma \\
&P_\sigma = (1/2q)\,(2B - \omega + \sigma q^2) \\
&\tilde{P}_\sigma = (1/2q)\,(\Omega - \omega + \sigma q^2) \\
&\xi_0 = \Omega - 2B
\end{aligned}
\tag{3.3}
$$

and the J_i are expressed as

$$
\begin{aligned}
\mathrm{Im}\, J_1(q,\omega) &= \frac{1}{|q|}\{-\tfrac{1}{4}(k_\uparrow^4 - P_\uparrow^4)\theta(k_\uparrow - |P_\uparrow|) \\
&\quad + [\tfrac{1}{4}(k_\downarrow^4 - P_\downarrow^4) + \tfrac{1}{2}(2B-\omega)(k_\downarrow^2 - P_\downarrow^2)]\theta(k_\downarrow - |P_\downarrow|)\} \\
\mathrm{Im}\, J_2(-q,\Omega-\omega) &= \frac{1}{|q|}\{-\tfrac{1}{4}(k_\downarrow^4 - \tilde{P}_\uparrow^4)\theta(k_\downarrow - |\tilde{P}_\uparrow|) \\
&\quad + [\tfrac{1}{4}(k_\downarrow^4 - \tilde{P}_\downarrow^4) + \tfrac{1}{2}(\Omega-\omega)(k_\downarrow^2 - \tilde{P}_\downarrow^2)]\theta(k_\downarrow - |\tilde{P}_\downarrow|)\} \\
\mathrm{Im}\, J_3(q,\omega) &= \frac{1}{|q|}\{\tfrac{1}{4}(k_\downarrow^4 - P_\downarrow^4)\theta(k_\downarrow - |P_\downarrow|) \\
&\quad + [-\tfrac{1}{4}(k_\uparrow^4 - P_\uparrow^4) + \tfrac{1}{2}(2B-\omega)(k_\uparrow^2 - |P_\uparrow|^2)]\theta(k_\uparrow - |P_\uparrow|)\} \\
\mathrm{Im}\, J_4(-q,\Omega-\omega) &= \frac{1}{|q|}\{\tfrac{1}{4}(k_\uparrow^4 - \tilde{P}_\downarrow^4)\theta(k_\uparrow - |\tilde{P}_\downarrow|) \\
&\quad + [-\tfrac{1}{4}(k_\uparrow^4 - \tilde{P}_\uparrow^4) + \tfrac{1}{2}(\Omega-\omega)(k_\uparrow^2 - \tilde{P}_\uparrow^2)]\theta(k_\uparrow - |\tilde{P}_\uparrow|)\}.
\end{aligned}
\tag{3.4}
$$

Here $n(|\omega|)$ denotes the Bose function, $\alpha = I\rho(\varepsilon_F)$, $\rho(\varepsilon_F)$ the density of states at the Fermi energy in the non-interacting system, f_0^{-+} and $f_{0\sigma}$ the reduced quantities of χ_0^{-+} and $\chi_{0\sigma}$ by $\rho(\varepsilon_F)$ respectively, λ_R and λ_I the real and imaginary parts of product $f_{0\uparrow}f_{0\downarrow}$ respectively,

$$k_\sigma = (1 + \sigma\zeta)^{1/3}$$

and q_c the cut-off momentum in the present electron-gas model. The relative magnetisation ζ is defined as $\zeta = M/N$, M the magnetisation and N the total number of electrons. Then the exchange field B is related to k_σ as

$$2B = k_\uparrow^2 - k_\downarrow^2.$$

Henceforth the arguments of f_0^{-+} and $f_{0\sigma}$, if they are omitted, are (q,ω) and $(-q,\Omega-\omega)$, respectively, and the temperature dependence of the Fermi function involved in the non-interacting susceptibilities is now neglected.

It is considered that the contributions in equation (3.2) are divided into two parts: the scattering processes and the decay processes of the spin wave. The former involves the spin wave and the latter the dissipative spin fluctuation as the transverse excitations in the final state.

The above expression is general for the long-wavelength spin wave within the present approximations; therefore, the spin-wave damping may be obtained as the function of Ω and T for any value of ζ. However, we only consider the two simple limits of weak and perfect ferromagnets in the following two sections. Hereafter we neglect the zero-point contribution, which comes from the T-independent terms within $N(\omega, \Omega)$ in equation (3.3), and approximate ξ_0 by $2B$.

4. Weakly ferromagnetic case

First we consider the weakly ferromagnetic limit in this section. In this limit $\zeta \ll 1$; therefore, the predominant region in q and ω space for the transverse excitation is for the dissipative spin fluctuation modes not for the spin waves, as is seen from figure 2 in appendix 1. The modes with small q and ω are strongly exchange-enhanced; therefore, the scattering process is also expected to give the dominant contribution for the damping especially in the low-temperature region.

The three η given in equation (3.2) are calculated by the use of f_0^{-+}, $f_{0\sigma}$ and $\mathrm{Im}\, J_i$ ($i = 1, \ldots, 4$) given in appendices 1 and 2, where each imaginary part of these is approximated by each value in the region III in equations (A1.12), (A1.15) and (A2.1)–(A2.4). Then the values of η are expressed as

$$
\begin{aligned}
\eta_1 \simeq & -\frac{27\alpha Q^2}{64\zeta^4}\int \mathrm{d}\omega\, N(\omega,\Omega) \\
& \times\left\{6\pi(\omega-\Omega)\theta(\omega-\omega_{\mathrm{D}})\theta(\omega_{\mathrm{C}}-\omega)\int_0^{q_1}\mathrm{d}q\, q\left(\frac{\zeta}{9\omega}\right)^{1/2}\delta\left[q-\left(\frac{9\omega}{\zeta}\right)^{1/2}\right]\right. \\
& \left. + C\omega(\omega-\Omega)\int_{q_1}^{q_c}\mathrm{d}q\,\frac{q^2}{A^2q^6+C^2\omega^2}\right\} \\
\eta_2 \simeq & -\frac{27\alpha^3(1+\alpha)^2CQ^2}{64\zeta^4}\int \mathrm{d}\omega\, N(\omega,\Omega)\omega(\omega-\Omega)\int_{q_1}^{q_c}\mathrm{d}q\,\frac{q^2}{g(q,\Omega-\omega)} \\
\eta_3 \simeq & \frac{81\alpha^2\Delta Q^2}{64\pi\zeta^4}\int \mathrm{d}\omega\, N(\omega,\Omega) \\
& \times\left\{\theta(\omega-\omega_{\mathrm{D}})\theta(\omega_{\mathrm{C}}-\omega)\int_0^{q_1}\mathrm{d}q\,\frac{3\pi Cq^3(\omega-\Omega)^2}{g(q,\Omega-\omega)}\left(\frac{\zeta}{9\omega}\right)^{1/2}\right. \\
& \left. \times\,\delta\left[q-\left(\frac{9\omega}{\zeta}\right)^{1/2}\right]-\int_{q_1}^{q_c}\mathrm{d}q\,\frac{(1-\alpha)AC^2q^8\omega(\omega-\Omega)}{(A^2q^6+C^2\omega^2)g(q,\Omega-\omega)}\right\}
\end{aligned}
\qquad (4.1)
$$

where

$$
\begin{aligned}
& g(q,\Omega-\omega) = (\Delta+2\alpha^2Aq^2)^2q^2+4\alpha^4C^2(\Omega-\omega)^2 \\
& \Delta = 1-\alpha^2+(G+F^2)\alpha^2\zeta^2
\end{aligned}
\qquad (4.2)
$$

and Δ has a positive value for ζ at low temperature, as is shown from the Hartree–Fock equilibrium condition at $T = 0$ for $\alpha \geqslant 1$, $\theta(\omega)$ the step function. The two cut-off frequencies involved in the contribution from the spin wave (i.e. $0 \leqslant q \leqslant q_1$), ω_{C} and

ω_D, are respectively upper and lower ones, and their expressions are given in appendix 3. The former comes from the crossing of the spin-wave dispersion with the Stoner boundary q_1 and the latter from the crossing with the lower boundary for the imaginary parts of $f_{0\sigma}$, J_2 and J_4, which is $-\bar{q}_1$ where q_1 and $\bar{q}_1$ are defined in appendix 1. The coefficients A, C, F and G are calculated from the band structure and are given in appendix 1 for the electron-gas model.

Equation (4.1) is evaluated for the contributions from the scattering process and from the decay process separately. The former contribution comes from the first terms in η_1 and η_3 and is evaluated for the three temperature regions. Retaining the most dominant contribution in each η, the final results are given as follows:

(i) $\omega_D \gg T$

$$[I\,\mathrm{Im}\,\delta\bar{\chi}^{-+}]_{\mathrm{sw}} \simeq \frac{81\pi\alpha\gamma_1}{16\zeta^4} Q^2T^2 + \frac{243\alpha^2\Delta C\gamma_3}{32\zeta^2}\frac{Q^2T^2}{\Omega} \tag{4.3}$$

(ii) $\Omega \gg T \gg \omega_D$

$$[I\,\mathrm{Im}\,\delta\bar{\chi}^{-+}]_{\mathrm{sw}} \simeq \frac{81\pi\alpha Q^2}{32\zeta^4}\left(\gamma_1 T^2 - \Omega T \ln\frac{\omega_D}{T}\right) + \frac{243\alpha^2\,\Delta C}{32\zeta^2}\frac{Q^2T^2}{\Omega} \tag{4.4}$$

(iii) $\omega_C \gg T \gg \Omega$

$$[I\,\mathrm{Im}\,\delta\bar{\chi}^{-+}]_{\mathrm{sw}} \simeq \frac{81\pi\alpha}{32\zeta^4} Q^2\Omega T\left(1 - \ln\frac{\omega_D}{T}\right) - \frac{243\alpha^2 C\gamma_1}{32\zeta^4\Delta} Q^2\Omega T^2 \tag{4.5}$$

where

$$\gamma_n = \int_0^\infty \mathrm{d}x\, h_n(x).$$

In these three expressions the first terms come from η_1 and the second ones from η_3.

Here we should give a comment on the logarithmic terms, which involve the cut-off frequency, ω_D. This cut-off frequency has its definite meaning at absolute zero temperature; therefore, if we take into account the temperature dependence of the Fermi function in the non-interacting susceptibilities, these terms are expected to disappear and to be supplanted by terms of the same order as or higher than the term that is independent of ω_D, through the smearing of the cut-off boundary. Furthermore if we consider the higher-temperature case, i.e. $T \gg \omega_C$, the effect of the smearing in not only ω_D but ω_C becomes important. Then the case is not considered here, but the dominant contribution is expected to be similar to (iii).

After ignoring the logarithmic terms, we compare the predominance of two terms in each temperature region. In both equations (4.3) and (4.4), the second terms are different from the first ones by order of ζ^4/Ω, because Δ is proportional to ζ^2. Therefore the dominant contribution comes from the first and second terms for $\zeta^4/\Omega \ll 1$ and $\zeta^4/\Omega \gg 1$, respectively. On the other hand, in equation (4.5) the second term is smaller than the first by a factor T/ζ^2 ($\ll T/\omega_C \ll 1$), because ω_C is proportional to ζ^3 (see appendix 3).

We next consider the contributions from the decay process in equation (4.1). By replacing the lower bound of q-integrals by $Q_c = q_1(\omega = 0)$, which is proportional to ζ, those contributions are given in the two limiting cases as follows:

(i) $Q_c^3 \gg T$

$$[I \operatorname{Im} \delta\bar{\chi}^{-+}]_{sf} \simeq \frac{27\alpha\gamma_1}{64A^2\zeta^4} Q^2\Omega T^2 \left(\frac{4C}{3Q_c^3} + \frac{(1+\alpha)^2 C}{\alpha^2}\kappa\right) \tag{4.6}$$

with

$$\kappa = \left(\frac{\alpha^2 A}{\Delta}\right)^{3/2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{2\alpha^2 A Q_c^2}{\Delta}\right)^{1/2}\right] - \frac{2\alpha^4 A^2 Q_c}{\Delta(2\alpha^2 A Q_c^2 + \Delta)}$$

(ii) $T \gg Q_c^3$

$$[I \operatorname{Im} \delta\bar{\chi}^{-+}]_{sf} \simeq \frac{27\pi\alpha Q^2\Omega T}{64A\zeta^4}\left(\ln\frac{q_c}{Q_c} + \frac{(l+\alpha)^2}{8\alpha^2}\ln\frac{2\alpha^2 A q_c^2}{2\alpha^2 A Q_c^2 + \Delta}\right). \tag{4.7}$$

In these expressions the two terms in the parentheses come from η_1 and η_2 in order. The contributions from η_3 are smaller than those by order of ζ^4, so are neglected in both cases. The present result, equation (4.6), in the low-temperature limit agrees with the result of Ma *et al* (1968) (see equation (4.6) in their paper) on the Q, Ω and T dependences.

Finally in this section we notice the importance of the scattering process in comparison with the decay process. In the low-temperature limit, comparing equation (4.6) and the first term of equation (4.3), the contribution from the decay process is smaller than that from the scattering process by order of Ω/ζ^3 ($\ll 1$). Even in the high-temperature limit the leading contributions of equations (4.5) and (4.7) have the same dependence of $\zeta^{-4}Q^2\Omega T$, though the latter is greater by order of $\ln(q_c/Q_c)$. Therefore the scattering process gives a significant contribution to the damping of the spin wave even in weak ferromagnets, not only at low temperature but also at high temperature. This is significantly different from the nearly ferromagnetic case under an external magnetic field (Ma *et al* 1968, Edwards and Fisher 1971).

5. Perfect ferromagnetic case

For the perfect ferromagnetic case, we put $\zeta = 1$ (i.e. $k_\uparrow = 2^{1/3}$ and $k_\downarrow = 0$); then $\chi_{0\downarrow}(q, \omega)$ or $f_{0\downarrow}(q, \omega)$ vanishes. Furthermore, for $\bar{T}(q, \omega)$, the dominant contribution is expected to be that from the spin-wave region in q- and ω-space because of the relative reduction of Stoner region in small q and ω region (see figure 2 in appendix 1) as long as Q and Ω are not too large. Then we only consider the contribution from the scattering process.

Then the three η of equation (3.2) are simplified as

$$\eta_1 \simeq \frac{4\pi\alpha Q^2}{3k_\uparrow^8}\int d\omega\, N(\omega, \Omega)\int_0^{q_1} dq\, q^2 \delta[f_0^{-+}(0,0) - \operatorname{Re} f_0^{-+}(q, \omega)] \operatorname{Im} J_4(-q, \Omega - \omega)$$

$$\eta_2 \simeq 0$$

$$\begin{aligned}\eta_3 \simeq \frac{2\alpha^2 Q^2}{3k_\uparrow^8}\int d\omega\, N(\omega, \Omega)\int_0^{q_1} dq\, \delta[f_0^{-+}(0,0) - \operatorname{Re} f_0^{-+}(q, \omega)] \\ \times \operatorname{Im} f_{0\uparrow}\{2(q^2 + \omega - \Omega)[-\tfrac{2}{3}k_\uparrow^3 + (q^2 + k_\uparrow^2 - \omega)\operatorname{Re} f_0^{-+}] \\ + \alpha[-\tfrac{2}{3}k_\uparrow^3 + (q^2 + k_\uparrow^2 - \omega)\operatorname{Re} f_0^{-+}]^2\}.\end{aligned} \tag{5.1}$$

These expressions are evaluated by the use of the expansion forms for $f_0^{-+}(q, \omega)$ and $f_{0\uparrow}(q, \omega)$ given in appendix 1 and of Im J_4 for the region III in equation (A2.4). Now we obtain the final results in three temperature limits as follows:

(i) $\omega_D \gg T$

$$[I \,\mathrm{Im}\, \chi^{-+}]_{\mathrm{sw}} \simeq \frac{5\pi\alpha\gamma_1}{2^{5/3}} Q^2T^2 + \frac{2\alpha^2 C}{3}(a_0 - 1)\gamma_1 Q^2 \Omega T^2 \tag{5.2}$$

(ii) $\Omega \gg T \gg \omega_D$

$$[I \,\mathrm{Im}\, \chi^{-+}]_{\mathrm{sw}} \simeq \frac{5\pi\alpha Q^2T^2}{2^{8/3}}\left(\gamma_1 - \frac{\Omega}{T}\ln\frac{\omega_D}{T}\right) + \frac{2\alpha^2 C}{3} Q^2\Omega T^2\left(2a_0\gamma_1 + \frac{\Omega}{T}\ln\frac{\omega_D}{T}\right) \tag{5.3}$$

(iii) $\omega_C \gg T \gg \Omega$

$$[I \,\mathrm{Im}\, \chi^{-+}]_{\mathrm{sw}} \simeq \frac{5\pi\alpha}{2^{8/3}} Q^2\Omega T\left(1 - \ln\frac{\omega_D}{T}\right) + \frac{2\alpha^2 C}{3} Q^2\Omega T^2\left(2a_0\gamma_1 + \frac{\Omega}{T}\ln\frac{\omega_D}{T}\right) \tag{5.4}$$

where

$$a_0 = 6 + \tfrac{4}{3}\alpha k_\uparrow$$

and the cut-off frequencies ω_C and ω_D are given in appendix 3.

In these expressions the first and second terms come from η_1 and η_3, respectively. As is noted in the previous section, the logarithmic terms are ignored below. In contrast to the weakly ferromagnetic case η_3 gives higher-order contributions than η_1 in all temperature regions.

Thus we obtain the dependence of Q^2T^2 for $\Omega \gg T$ and of $Q^2\Omega T$ for $T \gg \Omega$, which are consistent with the results of Gergis (1972).

6. Conclusions and discussion

The damping of the spin wave in itinerant electron ferromagnets has been calculated for both limits of weak and perfect ferromagnets.

Even in weak ferromagnets, it is shown that the scattering process of the spin wave with the longitudinal fluctuation gives a dominant contribution to the damping over the decay process especially at low temperature.

The scattering process yields damping proportional to Q^2T^2/ζ^4 for $\zeta^4 \ll \Omega$ and Q^2T^2/Ω for $\zeta^4 \gg \Omega$ at low temperature and proportional to $Q^2\Omega T/\zeta^4$ at high temperature. On the other hand the decay process gives damping proportional to $Q^2\Omega T^2/\zeta^7$ at low temperature and proportional to $(Q^2\Omega T/\zeta^4)\ln(q_c/Q_c)$ at high temperature. In comparison between these two processes, it is concluded that the scattering process is predominant over the decay process by order of ζ^3/Ω ($\gg 1$) for $\zeta^4 \ll \Omega$ and ζ^7/Ω^2 for $\zeta^4 \gg \Omega$ at low temperature. At high temperature both processes have the same $Q^2\Omega T/\zeta^4$ dependence, though the latter process is greater than the former by the factor of $\ln(q_c/Q_c)$.

This ratio between the two processes seems to be reasonable from a physical point of view. At low temperature, as the energy of the referred spin wave Ω becomes small relative to ω_c ($\propto \zeta^3$), the scattering process becomes more important because spin waves with long wavelength and low energy are mainly excited. On the other hand, at high temperature the dominant process depends on the ratio of the allowed region for the spin wave to that for the transverse dissipative fluctuation, that is q_c/Q_c, in the wavevector space.

The paramagnon theory by Ma *et al* (1968) gives a term proportional to $Q^2\Omega T^2$, which corresponds to the result at low temperature from the decay process. In their situation under a magnetic field, the spin-wave dispersion has a gap proportional to the external magnetic field, leading to the restriction of the scattering process. However, in *real* weak ferromagnets, the spin wave with sufficiently low energy may be enhanced; therefore the scattering process gives the dominant contribution, as is shown above, contrary to the expectation by Edwards and Fisher (1971).

For perfect ferromagnets, only the scattering process, which is expected to be predominant, has been calculated. In this case the damping is proportional to Q^2T^2 at low temperature and to $Q^2\Omega T$ at high temperature. This dependence is the same as the results by Gergis (1972) if we adopt the relation $\Omega \propto Q^2$ from the spin-wave dispersion.

In comparison of weak and perfect ferromagnets for the scattering process, it is shown that the damping in weak ferromagnets is greater than in perfect ones by order of ζ^{-4} at any temperature.

The intrinsic damping proportional to Q^6 proposed by Thompson (1965) is not derived in the present theory. This temperature-independent contribution is expected to occur if the zero-point contributions are taken into account.

Next we discuss the effect of electron–electron scattering, which brings the self-energy into the one-particle Green function in equation (2.5) and leads to the damping of the spin wave. For the sake of taking this effect into account in a simple manner, the *t*-matrix for the interaction between the same two species of particles each with opposite spin should be considered. However, the particle–particle *t*-matrix only gives a weak temperature dependence due to a Fermi function different from the electron–hole *t*-matrix. Therefore, this effect should be important only at sufficiently low temperature. The theory including this effect should be developed in future.

Finally, the present results are restricted for the limits of weak and perfect ferromagnets, but in future the intermediate case should be investigated and then comparison with the experimental results will be possible.

Acknowledgments

I would like to thank Professor T Moriya, Drs Y Takahashi and K Ueda for their valuable discussions. This work was performed using facilities of the Institute for Solid State Physics, University of Tokyo.

Appendix 1. Non-interacting susceptibilities

The non-interacting susceptibilities under the exchange field B are given by

$$\begin{aligned}\chi_0^{-+}(q,z) &= \rho(\varepsilon_F) f_0^{-+}(q,z) = \sum_k \frac{f_{k+q\downarrow} - f_{k\uparrow}}{z + \varepsilon_k - \varepsilon_{k+q} - 2B} \\ \chi_{0\sigma}(q,z) &= \rho(\varepsilon_F) f_{0\sigma}(q,z) = \sum_k \frac{f_{k+q\sigma} - f_{k\sigma}}{z + \varepsilon_k - \varepsilon_{k+q}}\end{aligned} \tag{A1.1}$$

where $z = \omega + is$ (s = positive infinitesimal) and $f_{k\sigma}$ is the Fermi function. The expressions for the electron-gas model at absolute zero temperature have already been given (Moriya and Kawabata 1973, Gumbs and Griffin 1976); therefore we only give the results. Using the reduced units given in the text and the electron-gas model, by neglecting the temperature dependence of the Fermi function we obtain

$$\operatorname{Re} f_0^{-+}(q,\omega) = \frac{1}{4q}\left(q(k_\uparrow + k_\downarrow) - \frac{\omega - 2B}{q}(k_\uparrow - k_\downarrow) + \frac{(q^2-q_1^2)(q^2-q_4^2)}{4q^2}\ln\left|\frac{(q-q_1)(q-q_4)}{(q+q_1)(q+q_4)}\right| + \frac{(q^2-q_2^2)(q^2-q_3^2)}{4q^2}\ln\left|\frac{(q-q_2)(q-q_3)}{(q+q_2)(q+q_3)}\right|\right) \tag{A1.2}$$

$$\operatorname{Im} f_0^{-+}(q,\omega) = (\pi/16|q|^3)\{(q^2-q_2^2)(q^2-q_3^2)\theta[(q_2^2-q^2)(q^2-q_3^2)] - (q^2-q_1^2)(q^2-q_4^2)\theta[(q_1^2-q^2)(q^2-q_4^2)]\} \tag{A1.3}$$

where

$$\left.\begin{matrix} q_1 \\ q_4 \end{matrix}\right\} = k_\uparrow \mp (k_\downarrow^2 + \omega)^{1/2} \qquad \left.\begin{matrix} q_2 \\ q_3 \end{matrix}\right\} = k_\downarrow \mp (k_\uparrow^2 - \omega)^{1/2} \tag{A1.4}$$

and

$$\operatorname{Re} f_{0\sigma}(q,\omega) = \frac{1}{4q}\left(2qk_\sigma + \frac{(q^2-\bar q_{1\sigma}^2)(q^2-\bar q_{4\sigma}^2)}{4q^2}\ln\left|\frac{(q-\bar q_{1\sigma})(q-\bar q_{4\sigma})}{(q+\bar q_{1\sigma})(q+\bar q_{4\sigma})}\right| + \frac{(q^2-\bar q_{2\sigma}^2)(q^2-\bar q_{3\sigma}^2)}{4q^2}\ln\left|\frac{(q-\bar q_{2\sigma})(q-\bar q_{3\sigma})}{(q+\bar q_{2\sigma})(q+\bar q_{3\sigma})}\right|\right) \tag{A1.5}$$

$$\operatorname{Im} f_{0\sigma}(q,\omega) = (\pi/16|q|^3)\{(q^2-\bar q_{2\sigma}^2)(q^2-\bar q_{3\sigma}^2)\theta[(\bar q_{2\sigma}^2-q^2)(q^2-\bar q_{3\sigma}^2)] - (q^2-\bar q_{1\sigma}^2)(q^2-\bar q_{4\sigma}^2)\theta[(\bar q_{1\sigma}^2-q^2)(q^2-\bar q_{4\sigma}^2)]\} \tag{A1.6}$$

where

$$\left.\begin{matrix} \bar q_{1\sigma} \\ \bar q_{4\sigma} \end{matrix}\right\} = k_\sigma \mp (k_\sigma^2 + \omega)^{1/2} \qquad \left.\begin{matrix} \bar q_{2\sigma} \\ \bar q_{3\sigma} \end{matrix}\right\} = k_\sigma \mp (k_\sigma^2 - \omega)^{1/2}. \tag{A1.7}$$

Therefore both $f_0^{-+}(q,\omega)$ and $f_{0\sigma}(q,\omega)$ have non-vanishing values of those imaginary parts in limited regions in (q,ω)-space. Those regions are illustrated in figures A1 and A2 for $f_0^{-+}(q,\omega)$ and $f_{0\sigma}(-q,\Omega-\omega)$ respectively, where the argument of $f_{0\sigma}$ is modified for convenience in the text.

Next we give the expansion forms of equations (A1.2) and (A1.5) for small q and ω and the explicit expressions of equations (A1.3) and (A1.6).

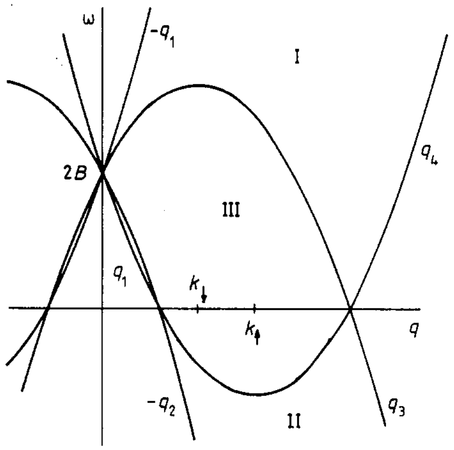

Figure A1. Non-vanishing regions for the imaginary part of the non-interacting transverse susceptibility, $f_0^{-+}(q, \omega)$, at $T = 0$. Each boundary line is given in equation (A1.4).

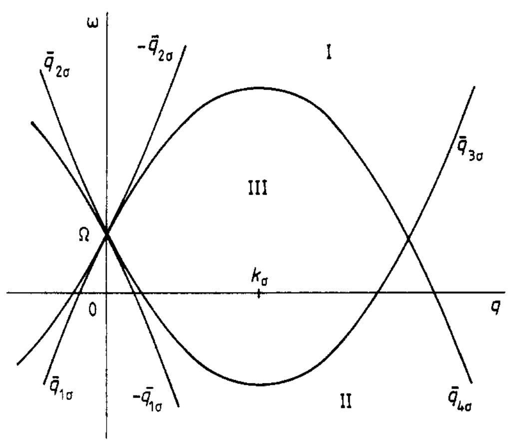

Figure A2. Non-vanishing regions for the imaginary part of the non-interacting longitudinal susceptibility, $f_{0\sigma}(-q, \Omega - \omega)$, at $T = 0$. Each boundary line is given in equation (A1.7).

(i) f_0^{-+}. For the real part of f_0^{-+}, we consider the spin-wave region ($q \leqslant q_1$) and the dissipative spin fluctuation region ($q \geqslant q_1$) separately. First, for $q \leqslant q_1$ we expand equation (A1.2) for small $q^2/2B$ and $\omega/2B$; then we obtain

$$f_0^{-+}(q, \omega) \simeq \left(\frac{1}{3B} + \frac{\omega}{6B^2}\right)(k_\uparrow^3 - k_\downarrow^3) - \frac{q^2}{6B^2}(k_\uparrow^3 + k_\downarrow^3) + \frac{q^2}{15B^3}(k_\uparrow^5 - k_\downarrow^5). \qquad \text{(A1.8)}$$

In this formula, B and k_σ are expanded by small ζ for weak ferromagnets and we put $k_\downarrow = 0$ for perfect ferromagnets. Then

$$\mathrm{Re}\, f_0^{-+}(q, \omega) \simeq 1 - \tfrac{1}{2}E\zeta^2 + \frac{3}{4\zeta}\omega - Aq^2 \qquad \text{(weak FM)} \qquad \text{(A1.9)}$$

$$\mathrm{Re}\, f_0^{-+}(q, \omega) \simeq \frac{2}{3k_\uparrow}(k_\uparrow^2 + \omega - \tfrac{1}{5}q^2) \qquad \text{(perfect FM).} \qquad \text{(A1.10)}$$

Next for $q > q_1$ only weak ferromagnets are considered and equation (A1.2) is expanded as

$$\mathrm{Re}\, f_0^{-+}(q, \omega) \simeq 1 - Aq^2 + D\frac{s}{q}\zeta - \tfrac{1}{2}E\zeta^2 \qquad \text{(weak FM)} \qquad \text{(A1.11)}$$

The imaginary part has finite value for $q > q_1$ and is given for the corresponding regions in figure A1 as follows:

$$\mathrm{Im}\, f_0^{-+}(q, \omega) = \begin{cases} (C/4q^3)[4q^2k_\uparrow^2 - (\omega - q^2 - 2B)^2] & \text{(for I)} \\ -(C/4q^3)[4q^2k_\downarrow^2 - (\omega + q^2 - 2B)^2] & \text{(for II)} \\ C\omega/q & \text{(for III).} \end{cases} \qquad \text{(A1.12)}$$

(ii) $f_{0\sigma}$. Here we give the expression for $f_{0\sigma}(-q, \Omega - \omega)$. The real part is expanded in the same way as f_0^{-+} for $q > q_1$ for weak ferromagnets and by small $q/k_\uparrow$ and $\omega/qk_\uparrow$ for perfect ferromagnets. The results are given as

$$\mathrm{Re}\, f_{0\sigma}(-q, \Omega - \omega) \simeq 1 - Aq^2 + \sigma F\zeta - \tfrac{1}{2}G\zeta^2 \qquad \text{(weak FM)} \qquad \text{(A1.13)}$$

$$\mathrm{Re}\, f_{0\sigma}(-q, \Omega - \omega) \simeq \begin{cases} k_\uparrow - (A/k_\uparrow)q^2 & (\sigma = \uparrow) \\ 0 & (\sigma = \downarrow) \end{cases} \quad \text{(perfect FM).} \qquad \text{(A1.14)}$$

The imaginary part of $f_{0\sigma}$ is given from equation (A1.6) for each region I, II and III in figure A2 as below. For the weak ferromagnets with $\sigma = \uparrow$ or $\downarrow$ and for perfect ferromagnets with $\sigma = \uparrow$,

$$\mathrm{Im}\, f_{0\sigma}(-q, \Omega - \omega) = \begin{cases} -(C/4q^3)[4q^2k_\sigma^2 - (\Omega - \omega + q^2)^2] & \text{(for I)} \\ (C/4q^3)[4q^2k_\sigma^2 - (\omega - \Omega + q^2)^2] & \text{(for II)} \\ C(\Omega - \omega)/q & \text{(for III)} \end{cases} \qquad \text{(A1.15)}$$

and for perfect ferromagnets with $\sigma = \downarrow$ this quantity vanishes.

The coefficients used above are

$$A = \tfrac{1}{12} \qquad C = \pi/4 \qquad D = \tfrac{1}{3} \qquad E = \tfrac{4}{27} \qquad F = \tfrac{1}{3} \qquad G = \tfrac{4}{9}$$

for the electron-gas model.

Appendix 2. Imaginary parts of J_1, J_2, J_3 and J_4

The explicit expression of the imaginary parts of the J_i in equation (3.4) is given in this appendix. The non-vanishing regions of the imaginary parts of J_1 and J_3 are the same as those of f_0^{-+} illustrated in figure A1. In the respective regions I, II and III in the figure, those are given by

$$\mathrm{Im}\, J_1(q, \omega) = \begin{cases} -\dfrac{1}{4q}\left[k_\uparrow^4 - \left(\dfrac{\omega - q^2 - 2B}{2q}\right)^4\right] & \text{(for I)} \\ \dfrac{1}{4q}\left[k_\downarrow^2 - \left(\dfrac{\omega + q^2 - 2B}{2q}\right)^2\right] & \\ \times\left[2k_\uparrow^2 - k_\downarrow^2 - 2\omega + \left(\dfrac{\omega + q^2 - 2B}{2q}\right)^2\right] & \text{(for II)} \\ -\dfrac{\omega}{4q}(2k_\uparrow^2 - \omega) & \text{(for III)} \end{cases} \qquad \text{(A2.1)}$$

$$
\operatorname{Im} J_3(q,\omega) = \begin{cases} \dfrac{1}{4q}\left[k_\uparrow^2 - \left(\dfrac{q^2-\omega+2B}{2q}\right)^2\right] \\ \quad \times\left[k_\uparrow^2 - 2k_\downarrow^2 - 2\omega - \left(\dfrac{q^2-\omega+2B}{2q}\right)^2\right] & \text{(for I)} \\ \dfrac{1}{4q}\left[k_\downarrow^4 - \left(\dfrac{q^2-2B+\omega}{2q}\right)^4\right] & \text{(for II)} \\ -\dfrac{\omega}{4q}(2k_\downarrow^2 + \omega) & \text{(for III).} \end{cases} \tag{A2.2}
$$

The non-vanishing regions of $\operatorname{Im} J_2(-q, \Omega - \omega)$ are the same as those of $f_{0\downarrow}(-q, \Omega - \omega)$ and the values are given by

$$
\operatorname{Im} J_2(-q,\Omega-\omega) = \begin{cases} -\dfrac{1}{4q}\left[k_\downarrow^4 - \left(\dfrac{\Omega-\omega+q^2}{2q}\right)^4\right] & \text{(for I)} \\ \dfrac{1}{4q}\left[k_\downarrow^2 - \left(\dfrac{\Omega-\omega-q^2}{2q}\right)^2\right] \\ \quad \times\left[2(\Omega-\omega) + k_\downarrow^2 + \left(\dfrac{\Omega-\omega-q^2}{2q}\right)^2\right] & \text{(for II)} \\ \dfrac{\Omega-\omega}{4q}(2k_\downarrow^2 + \Omega - \omega) & \text{(for III)} \end{cases} \tag{A2.3}
$$

for the three regions in figure A2. Similarly the non-vanishing regions of $\operatorname{Im} J_4(-q, \Omega - \omega)$ correspond with $\operatorname{Im} f_{0\uparrow}(-q, \Omega - \omega)$ and are expressed as

$$
\operatorname{Im} J_4(-q,\Omega-\omega) = \begin{cases} \dfrac{1}{4q}\left[k_\uparrow^2 - \left(\dfrac{\Omega-\omega+q^2}{2q}\right)^2\right] \\ \quad \times\left[2(\Omega-\omega) - k_\uparrow^2 - \left(\dfrac{\Omega-\omega+q^2}{2q}\right)^2\right] & \text{(for I)} \\ \dfrac{1}{4q}\left[k_\uparrow^4 - \left(\dfrac{\Omega-\omega-q^2}{2q}\right)^4\right] & \text{(for II)} \\ \dfrac{\Omega-\omega}{4q}(2k_\uparrow^2 - \Omega + \omega) & \text{(for III).} \end{cases} \tag{A2.4}
$$

Appendix 3. RPA spin wave and cut-off energies

First we derive a spin-wave dispersion in RPA within the electron-gas model. This is derived from the pole of $\bar{\chi}^{-+}(q, \omega)$ defined in equation (2.10) in the vanishing region of $\operatorname{Im} \chi_0^{-+}$, that is

$$1 - I\,\mathrm{Re}\,\chi_0^{-+}(q,\omega) = 0. \tag{A3.1}$$

With the use of the value at $q = 0$ and $\omega = 0$, this equation is rewritten as

$$f_0^{-+}(0,0) - \mathrm{Re}\,f_0^{-+}(q,\omega) = 0 \tag{A3.2}$$

and is expressed for the long-wavelength spin wave as

$$\omega \simeq D_{\mathrm{sw}} q^2$$
$$D_{\mathrm{sw}} \simeq \frac{1}{k_\uparrow^3 - k_\downarrow^3}\left(k_\uparrow^3 + k_\downarrow^3 - \frac{2}{5B}(k_\uparrow^5 - k_\downarrow^5)\right) \tag{A3.3}$$

by using equation (A1.8). The spin-wave stiffness constant D_{sw} is rewritten by using equations (A1.9) and (A1.10) as

$$D_{\mathrm{sw}} \simeq \begin{cases} \frac{1}{9}\zeta & \text{(weak FM)} \\ \frac{1}{5} & \text{(perfect FM).} \end{cases} \tag{A3.4}$$

Next we calculate the upper cut-off frequency ω_{C}, which is given by the intersection of the spin-wave dispersion and the Stoner boundary q_1. Therefore by solving equations (A3.3), (A3.4) and q_1 in equation (A1.4), we obtain

$$\omega_{\mathrm{C}} \simeq \begin{cases} \frac{4}{81}\zeta^3 & \text{(weak FM)} \\ k_\uparrow^2(3-\sqrt{5})/8 & \text{(perfect FM).} \end{cases} \tag{A3.5}$$

Lastly, the lower cut-off frequency ω_{D} is defined by the intersection of the spin-wave dispersion and $-\bar{q}_{1\sigma}(\Omega - \omega)$ given in equation (A1.7) and figure A2. Therefore ω_{D} is spin-dependent. However, in the lowest-order term for small ζ and Ω in weak ferromagnets, it has no spin dependence and is given by

$$\omega_{\mathrm{D}} \simeq \begin{cases} \frac{1}{36}\zeta\Omega^2 & \text{(weak FM)} \\ (1/20\,k_\uparrow^2)\Omega^2 & \text{(perfect FM).} \end{cases} \tag{A3.6}$$

References

Edwards D M and Fisher B 1971 *J. Physique Coll.* **32** (C1) 697
Gergis I S 1972 *Int. J. Magn.* **3** 273
Gumbs G and Griffin A 1976 *Phys. Rev.* B **13** 5054
Hertz J A 1971 *Int. J. Magn.* **1** 253
Kawabata K 1974 *J. Phys. F: Met. Phys.* **4** 1477
Ma S, Béal-Monod M T and Fredkin D R 1968 *Phys. Rev.* **174** 227
Moriya T 1976 *J. Phys. Soc. Japan* **40** 933
Moriya T and Kawabata A 1973 *J. Phys. Soc. Japan* **34** 639
Onodera S, Ishikawa Y and Tajima K 1981 *J. Phys. Soc. Japan* **50** 1513
Stringfellow M W 1968 *J. Phys. C: Solid State Phys.* **1** 950
Thompson E D 1965 *J. Appl. Phys.* **36** 1133
Yamada H 1976 *J. Phys. Soc. Japan* **41** 753